The Spectral Radius of A Planar Graph

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ABSTRACT

A decomposition result for planar graphs is used to prove that the spectral radius of a planar graph on n vertices is less than $4 + \sqrt{3(n-3)}$. Moreover, the spectral radius of an outerplanar graph on n vertices is less than $1 + \sqrt{2 + \sqrt{2}} + \sqrt{n-5}$.

1. INTRODUCTION

All graphs are finite, undirected, without loops and multiple edges. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The *complement* in G of a subgraph H is the subgraph of G obtained by deleting all edges in H. The *join* $G_1 \vee G_2$ of two graphs G_1 and G_2 is obtained by adding an edge from each vertex in G_1 to each vertex in G_2. Let K_n be the complete graph and P_n the path with n vertices. Let $\delta(G)$ and $\Delta(G)$ be the minimum and the maximum degree of vertices in G. The spectral radius $r(G)$ of G is the largest eigenvalue of its adjacency matrix $A(G)$.

Spectra of graphs have been studied in recent years, but the results are often weak when applied to planar graphs. In 1978, A. J. Schwenk and R. J. Wilson [6] asked, in particular, what can be said about the eigenvalues of a planar graph. For ten years after this paper little work was done on this problem. Then in 1988 Hong Yuan [7] proved that the spectral radius of a planar graph on n vertices is less than or equal to $\sqrt{5n-11}$. In [3] Cvetković and Rowlinson conjectured that $K_1 \vee P_{n-1}$, with spectral radius very close to $1 + \sqrt{n}$, is the unique graph with the largest spectral radius among all

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outerplanar graphs with n vertices. In this paper we improve some decomposition results for planar graphs to show that $r(G) < 4 + \sqrt{3(n-3)}$ for any planar graph G on n vertices. Moreover, $r(G) < 1 + \sqrt{2+\sqrt{2}} + \sqrt{n-5}$ for any outerplanar graph on n vertices.

2. RESULTS

LEMMA 1 (Courant-Weyl inequality) [4]. *Let A and B be two real symmetric matrices of order n. If $C = A + B$ then $r(C) \leqslant r(A) + r(B)$.*

LEMMA 2 (Hong Yuan) [7]. *If G is a graph with n vertices, m edges, and no isolated vertices, then*

$$r(G) \leqslant \sqrt{2m - n + 1}$$

with equality if and only if G is the disjoint union of either a star or a complete graph with copies of K_2.

LEMMA 3 (Barnette) [1]. *If G is planar and 3-connected, then G has a spanning tree with maximum degree at most 3.*

The following lemma is easy to prove.

LEMMA 4. *Let T be a tree with at least one vertex of degree 3. Color some vertices of degree 3 red. Then there exists a red vertex v with at least two neighbors that are not adjacent to any other red vertices.*

LEMMA 5. *A maximal planar graph with at least four vertices has a disjoint edge decomposition into a spanning tree with maximum degree at most 4 and a spanning subgraph with no isolated vertices.*

Proof. For a spanning tree T of the graph G let $\bar{T}$ denote the complement of T in G. Color the isolated vertices of $\bar{T}$ red. Consider the set **S** of all spanning trees T of G with the following properties:

(1) $\Delta(T) \leqslant 4$.
(2) All red vertices have degree 3 in G.
(3) For any vertex v, if $\deg_T(v) = 4$ then v is adjacent to no red vertex.

We first claim that **S** is not empty. Since G is maximal, G is 3-connected, and by Lemma 3, G has a spanning tree T with $\Delta(T) \leqslant 3$. This verifies conditions (1) and (3). Since G is 3-connected, a red vertex v must satisfy $\deg_T(v) = \deg_G(v) = 3$, verifying condition (2).

Let T be a spanning tree in $\mathbf{S}$ such that $\overline{T}$ has the least number of isolated vertices. If $\overline{T}$ has no isolated vertex, we are done. Otherwise, by Lemma 4 there exists a red vertex u with at least two neighbors that are not adjacent to other red vertices. Construct a new tree T' as follows by edge switching on a single K_4. Let x, y, z be the three vertices adjacent to u where x and y are not adjacent to other red vertices. Then the edges $ux, uy, uz \in T$ and $xy, yz, zx \in \overline{T}$. Now let $T' = T - ux + xy$. Clearly T' is a spanning tree of G. We claim that conditions (1)–(3) hold for T'. Only vertex y has increased its degree in the spanning tree. Since y was adjacent to red vertex u, by condition (3) we have $\deg_{T'}(y) = \deg_T(y) + 1 \leqslant 3 + 1 = 4$. Hence $\Delta(T') \leqslant 4$, verifying condition (1). Since $ux, zy \in \overline{T'}$, vertices u, x, y, and z are not isolated in $\overline{T'}$. In particular, vertex u is no longer red in $\overline{T'}$. Hence the the set of isolated vertices of $\overline{T'}$ is a proper subset of the set of isolated vertices of $\overline{T}$. This verifies that condition (2) holds for T', and also condition (3), because vertex y, the only possible vertex of degree 4 in T' not of degree 4 in T, was adjacent to no red vertex besides u.

The fact that the complement of T' in G has one less isolated vertex than the complement of T contradicts the minimality of the number of isolated vertices. ∎

THEOREM 1. *If G is a planar graph with $n \geqslant 3$ vertices, then*

$$r(G) < 4 + \sqrt{3(n-3)}\,.$$

Proof. Suppose $n \geqslant 4$; otherwise the result is obvious. Let G' be a maximal planar graph containing G as a spanning subgraph. By Lemma 5, G' can be decomposed into a spanning tree T with maximum degree at most 4 and a spanning subgraph H with no isolated vertex. Then $A(G') = A(T) + A(H)$ implies, by Lemma 1, that $r(G') \leqslant r(T) + r(H)$. Since the largest eigenvalue of a graph is less than or equal to the maximum degree with equality if and only if the graph is regular [2], then $r(T) < \Delta(T) \leqslant 4$. Since H has $(3n-6) - (n-1) = 2n-5$ edges and no isolated vertices, by Lemma 2, $r(H) \leqslant \sqrt{2(2n-5) - n + 1} = \sqrt{3(n-3)}$. Therefore $r(G) \leqslant r(G') < 4 + \sqrt{3(n-3)}$. ∎

It is not difficult to show that $r(\overline{K_2} \vee C_{n-2}) = 1 + \sqrt{2n-3}$, and that $r(P_2 \vee P_{n-2})$ is between $1 + \sqrt{2n-3}$ and $2 + \sqrt{2n-3}$. After examining the spectral radius of several other families of graphs and some small graphs we conjecture that $P_2 \vee P_{n-2}$ and $\overline{K_2} \vee C_{n-2}$ are optimal in the following sense:

CONJECTURE 1. If G is a planar graph with n vertices, then

$$r(G) \leqslant r(P_2 \vee P_{n-2}) < 2 + \sqrt{2n-3}.$$

Moreover, if $\delta(G) = 4$ then $r(G) \leqslant r(\overline{K_2} \vee C_{n-2}) = 1 + \sqrt{2n-3}$.

REMARK. To prove the conjecture it is tempting to try to find a spanning subgraph of G with small maximum degree, such that its complement in G has about $3n/2$ edges and no isolated vertices. The same argument used in Theorem 1 would then improve the bound of Theorem 1 from order of magnitude $\sqrt{3n}$ to order $\sqrt{2n}$. However, the graph $P_2 \vee P_{n-2}$ allows no such decomposition. Another approach is required if the conjecture is true.

3. OUTERPLANAR GRAPHS

An outerplanar graph G is a graph which can be embedded in the plane so that all vertices are on one face, say the outer face. An *internal triangle* is a triangle with no edge on the outer face. Let U_n $(n \geqslant 4)$ be the set of all such maximal outerplanar graphs which have n vertices and no internal triangles. Rowlinson [5] proved that $K_1 \vee P_{n-1}$ is the unique graph in U_n with maximal spectral radius. Moreover, he and Cvetković [3] conjectured that $K_1 \vee P_{n-1}$ is the unique graph with maximal spectral radius among all outerplanar graphs with n vertices. It is not difficult to prove that the largest eigenvalue of $K_1 \vee P_{n-1}$ is between $1 + \sqrt{n} - 2/(2 + n - 2\sqrt{n})$ and $1 + \sqrt{n}$. The theorem in this section comes close to confirming the conjecture of Rowlinson and Cvetković.

Clearly, a maximal outerplanar graph can be decomposed into a spanning 2-regular subgraph, the outer face, and its complement in G with exactly $n - 3$ edges. Furthermore, we have the following improvement.

LEMMA 6. *A maximal outerplanar graph G has a spanning subgraph H with the following properties*:

(1) $\Delta(H) \leqslant 4$.

(2) *The complement of H in G has no isolated vertices.*

(3) *H consists of a single cycle together with some pendant edges.*

(4) *No two vertices of degree 4 in H are adjacent.*

(5) *If H has a vertex with degree 4, then H also has a vertex with degree 2.*

Proof. If the order n of G is less than 7, then Lemma 6 can be routinely checked. Suppose that $n \geqslant 7$. Let $C = (v_1 v_2 \cdots v_n)$ be the Hamiltonian cycle of G on the outer face, and $D = \{v_{k_1}, v_{k_2}, \ldots, v_{k_s}\}$ the set of all vertices of degree 2 in G. It is easy to prove that $|D| \geqslant 2$ and no two vertices of D are adjacent in G. Let $v_i \in D$. By the maximality of G, v_{i-1} must be adjacent to v_{i+1}, with the convention $v_{n+1} = v_1$. Further, either v_{i+1} or v_{i-1} has degree at least 4. Let F be the subgraph of G obtained from C by adding the set of edges $\{v_{k_1-1}v_{k_1+1}, v_{k_2-1}v_{k_2+1}, \ldots, v_{k_s-1}v_{k_s+1}\}$. Further, let H be the subgraph of G obtained from F by the following procedure. For each $v_{k_i} \in D$ let u and y be the two neighbors of v_{k_i}. There are three cases:

Case 1. If one of the neighbors of v_{k_i}, say u, has degree 3, then delete the edge uv_{k_i} from F.

Case 2. If one of the neighbors of v_{k_i}, say u, has degree 4 and is adjacent to another vertex v in D, then delete the edges uv_{k_i} and uv from F. Since G is maximal, the second neighbor x of v is adjacent to y. Since $n \geqslant 7$, the degrees of x and y are at least 4; moreover, neither x nor y can assume the role of u, and so neither of the edges xv or yv_{k_i} is deleted from F. (In Figure 1 the dark arcs are in F and the dotted arcs are those deleted from F.)

Case 3. Otherwise delete $v_{k_i}u$ from F, where u is either neighbor of v_{k_i}.

Now the remaining graph H is clearly a spanning subgraph satisfying properties (1), (2), and (3). Concerning property (4), assume vertices x and z are adjacent in H and both have degree 4 in H. Then x and z also have degree 4 in F, which implies that x and z are both adjacent to a vertex y

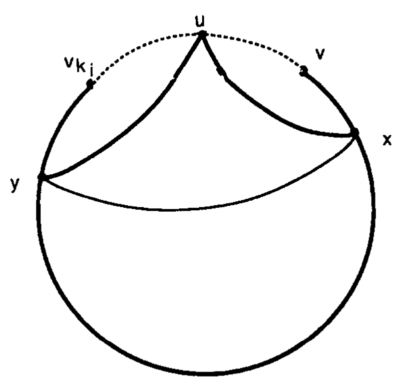

FIG. 1. Case (2) of Lemma 6.

with degree 2 in F. But by the construction, either xy or zy was deleted from F in forming H, making it impossible for both x and y to have degree 4. This is a contradiction. Concerning property (5), note that the pendant vertices of H are the vertices of D. Let Q denote the set of vertices on the cycle of H (nonpendant vertices). Since $|D| \leqslant \frac{1}{2}n$, therefore $|Q| \geqslant \frac{1}{2}n$. If there are no vertices of degree 2 and at least one vertex of degree 4 in Q, then the total number of vertices in H is greater than $2(n/2) + 1 = n + 1$, a contradiction. ∎

REMARK. Note that if $G = K_1 \vee P_{n-1}$, then G does not have a spanning, unicyclic, connected subgraph with maximum degree 3 such that its complement in G has no isolated vertices. In this sense Lemma 6 cannot be improved.

LEMMA 7. *If H is the graph in Lemma* 6, *then*

$$r(H) \leqslant 1 + \sqrt{2 + \sqrt{2}} < 3.$$

Proof. Let L_5 be the graph in Figure 2. We claim that H can be decomposed into an edge disjoint union of subgraphs I and J where each component of I is isomorphic to K_2 and each component of J is isomorphic to a subgraph of P_5 or L_5.

To see this let $C = (u_1, u_2, \dots, u_s)$ be the cycle in H. If s is even, let I consist of every other edge in C and let J consist of all edges in H not in I. Properties (3) and (4) in Lemma 6 guarantee that the claim is true.

If s is odd, let I' consist of every other edge in C beginning at a vertex of degree 2, if such a vertex exists. Let J' consist of all edges in H not in I'. Then all components of $C \setminus I'$ are isomorphic to K_2 except one that is isomorphic to P_3. Let the three consecutive vertices of the P_3 component mentioned above be x, y, z. If y is adjacent to a vertex v other than x or z in H, then let $I = I' + yv$ and $J = J' - yv$. Otherwise let $I = I'$ and $J = J'$. If H has no vertices of degree 4, then each component of J is

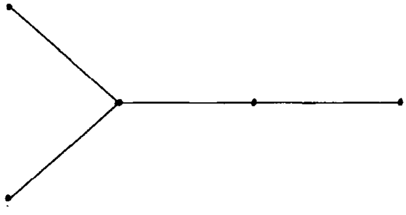

FIG. 2. The graph L_5.

isomorphic to a subgraph of P_5. If there is a vertex of degree 4 in H, then by property (5) of Lemma 6, the graph H has a vertex with degree 2, which without loss of generality can be chosen to be vertex x. Then each component of J is isomorphic to a subgraph of L_5. This verifies the claim in the odd case.

Since $r(I) = 1$ and $r(J) = \max\{r(P_5), r(L_5)\} = r(L_5) = \sqrt{2+\sqrt{2}}$, by Lemma 1 we have $r(H) \leqslant r(I) + r(J) \leqslant 1 + \sqrt{2+\sqrt{2}} < 3$. ■

THEOREM 2. *If G is an outerplanar graph with n vertices, then*

$$r(G) \leqslant 1 + \sqrt{2+\sqrt{2}} + \sqrt{n-5}.$$

Proof. Let G' be a maximal outerplanar graph containing G as a spanning graph. Let H be the unicyclic subgraph of Lemma 6, and F its complement in G'. Since F has no isolated vertices and $(2n-3) - n = n - 3$ edges, Lemma 2 yields $r(F) \leqslant \sqrt{2(n-3) - n + 1} = \sqrt{n-5}$. By Lemmas 1 and 7, $r(G) \leqslant 1 + \sqrt{2+\sqrt{2}} + \sqrt{n-5}$.

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